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*Publication date:*  
2009

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*

Frendrup, A., Hartnell, B., & Vestergaard, P. D. (2009). *A note on equimatchable graphs*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2009-07

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by

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R-2009-07

April 2009

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# A note on equimatchable graphs

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## Abstract

Let  $G = (V, E)$  be a graph. A set  $M$  of edges is called a matching in  $G$  if each vertex in  $G$  belongs to at most one edge from  $M$ , and  $M$  is a maximal matching if any edgeset  $M'$ , such that  $M \subset M'$ , is not a matching in  $G$ . If all maximal matchings in  $G$  have the same cardinality then  $G$  is an equimatchable graph. In this paper we characterize the equimatchable graphs of girth at least five. As a consequence we also determine those graphs of girth five or more in which every minimal set of edges dominating edges is a minimum.

**Keywords:** equimatchable, maximal matching

**AMS subject classification:** 05C69

# 1 Introduction

For notation and graph theory terminology we in general follow [5]. Specifically, let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*. The *girth* of a graph  $G$ , denoted  $g(G)$ , is the length of the shortest cycle or circuit of  $G$ .

A set  $M$  of edges is called a *matching* in  $G$  if each vertex in  $G$  belongs to at most one edge in  $M$ , and  $M$  is a *maximal matching* if any edgeset  $M'$ , such that  $M'$  contains  $M$  as a proper subset,  $M \subset M'$ , is not a matching in  $G$ . If all maximal matchings in  $G$  have the same cardinality then  $G$  is a *equimatchable graph*. For a matching  $M$  we define  $A(M)$  to be all the edges in  $M$  or incident with an edge from  $M$  or in  $M$ .

Although the equimatchable graphs were characterized in [1] they are not explicitly described. In fact, as stated in [1] "using our characterization we show that membership in this class can be polynomially determined." In this note, we give an explicit, easy to recognize, description of such graphs in the case that the girth is 5 or more. Part of the motivation is also the connection to three similar problems. Graphs in which every minimal set of dominating vertices is of one size (called well-dominated [3]), graphs in which every minimal set of vertices dominating edges is of fixed size (the well-covered graphs [2]) and graphs in which every minimal set of edges dominating vertices (uniform size star factors [4]) have all been characterized in the situation that the girth is five or more. This leaves the case of graphs in which every minimal set of edges dominating edges is of one cardinality (call these graphs Class A). Noting that a maximal matching is also a minimal edge dominating set, it follows that the equimatchable graphs contain the class A ones as a sub-collection. In particular, if a graph is in the class A collection then all independent as well as non-independent minimal dominating sets are of one cardinality whereas a graph that is equimatchable only requires the independent minimal dominating sets to be equal. As it turns out, for girth at least 5, the two collections are identical while at girth 4 the class A ones are a proper subcollection of the equimatchable ones.

## 2 Results on equimatchable graphs

We begin by stating some observations on matchings and equimatchable graphs. In addition a number of lemmas are established that will be useful in proving the main result in this note.

**Observation 1** *If  $M$  is a matching in a graph  $G$ , then  $M$  can be extended to a maximal matching  $M'$ . That is,  $M \subseteq M'$ .*

**Observation 2** *A graph  $G$  is equimatchable if and only if each component of  $G$  is equimatchable.*

**Observation 3** *If  $M$  is a matching in a equimatchable graph  $G$ , then  $G - A(M)$  is a equimatchable graph.*

**Lemma 1** *Let  $G$  be a equimatchable graph. If  $M_1$  and  $M_2$  are matchings in  $G$  and  $A(M_1) \subseteq A(M_2)$ , then  $|M_1| \leq |M_2|$ .*

**Proof.** Let  $M'_2$  be a maximal matching such that  $M_2 \subseteq M'_2$ . Since  $A(M_1) \subseteq A(M_2)$  the set  $M_1 \cup (M'_2 \setminus M_2)$  is a matching in  $G$ . Thus there is a maximal matching  $M'_1$  such that  $M_1 \cup (M'_2 \setminus M_2) \subseteq M'_1$ . If  $G$  is equimatchable then  $|M'_2| = |M'_1| \geq |M_1| + |M'_2 \setminus M_2| = |M_1| - |M_2| + |M'_2|$  and therefore  $|M_1| \leq |M_2|$ .  $\square$

**Lemma 2** *If  $G \cong C_k$  then  $G$  is equimatchable if and only if  $k \in \{3, 4, 5, 7\}$ .*

**Proof.** Let  $G \cong C_k : v_1, v_2, \dots, v_k, v_1$ . If  $k \leq 7$  the statement can easily be checked. Assume that  $k \geq 8$  and let  $M_1 := \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$  and  $M_2 := \{v_1v_2, v_4v_5, v_7v_8\}$ . Then  $A(M_1) = A(M_2)$  and Lemma 1 implies that  $|M_1| = |M_2|$  if  $G$  is equimatchable. Since  $|M_1| \neq |M_2|$  this implies that  $G$  is not equimatchable.  $\square$

**Lemma 3** *Let  $G \not\cong K_2$  be a connected equimatchable graph. Then  $G$  does not contain a path  $v_1, v_2, \dots, v_{2k}$  such that  $v_1$  and  $v_{2k}$  are stems.*

**Proof.** Assume that  $G$  contains a path  $v_1, v_2, \dots, v_{2k}$  such that  $v_1$  and  $v_{2k}$  are stems and let  $l_1$  be a leaf adjacent to  $v_1$  and let  $l_2$  be a leaf adjacent to  $v_{2k}$ . If  $M_1 := v_1v_2, v_3v_4, \dots, v_{2k-1}v_{2k}$  and  $M_2 := l_1v_1, v_2v_3, \dots, v_{2k}l_2$  then  $M_1$  and  $M_2$  are matchings in  $G$  and  $A(M_1) = A(M_2)$ . From Lemma 1 it follows that  $|M_1| = |M_2|$  but by construction  $|M_2| = |M_1| + 1$ . Since this is a contradiction the statement is verified.  $\square$

**Lemma 4** *Let  $G$  be a equimatchable graph with  $g(G) \geq 8$ . Then  $G$  does not contain a path  $v_1, v_2, v_3, v_4$  such that neither  $v_1$  nor  $v_4$  are stems.*

**Proof.** Assume otherwise and let  $P : v_1, v_2, v_3, v_4$  be a path such that neither  $v_1$  nor  $v_4$  are stems. For each edge  $e$  incident to  $v_1$  or  $v_4$  not on  $P$  let  $e'$  be an edge incident with  $e$  where  $e'$  does not include a vertex of  $P$ . Let  $M$  be the set containing all of these edges. Then  $M$  is a matching and  $A(M) \cap \{v_1v_2, v_2v_3, v_3v_4\} = \emptyset$ . Thus  $M_1 := M \cup \{v_1v_2, v_3v_4\}$  and  $M_2 := M \cup \{v_2v_3\}$  are matchings in  $G$ . By construction  $A(M_1) = A(M_2)$  and thus Lemma 1 implies that  $|M_1| = |M_2|$  but this contradicts the fact that  $|M_1| = 1 + |M_2|$ .  $\square$

**Lemma 5** *Let  $G$  be a equimatchable graph with  $g(G) \geq 5$ . If  $G$  contains a path  $v_1, v_2, v_3$  such that  $v_1$  is a stem then  $v_3$  is also a stem.*

**Proof.** Assume otherwise and let  $P : v_1, v_2, v_3$  be a path such that  $v_1$  is a stem and  $v_3$  is not a stem. Let  $l$  be a leaf adjacent to  $v_1$ . For each edge  $e$  incident to  $v_3$  not on  $P$  let  $e'$  be

an edge incident with  $e$  but not incident with  $v_3$ . Let  $M$  be the set containing all of these edges. Then  $M$  is a matching and  $A(M) \cap \{lv_1, v_1v_2, v_2v_3\} = \emptyset$ . Thus  $M_1 := M \cup \{lv_1, v_2v_3\}$  and  $M_2 := M \cup \{v_1v_2\}$  are matchings in  $G$ . By construction  $A(M_1) = A(M_2)$  and thus Lemma 1 implies that  $|M_1| = |M_2|$  but this contradicts the fact that  $|M_1| = 1 + |M_2|$ .  $\square$

Let  $\mathcal{F}$  be the family of graphs containing  $K_2$  and all connected bipartite graphs  $G$  with bipartite sets  $V_1$  and  $V_2$  such that all vertices in  $V_1$  are stems and no vertex from  $V_2$  is a stem.

**Lemma 6** *Each graph from  $\mathcal{F}$  is equimatchable.*

**Proof.** Consider a graph  $G$  from  $\mathcal{F}$ . If  $M$  is a maximal matching in  $G$  then for each edge  $e = uv$  between a stem  $u$  and a leaf  $v$  the matching  $M$  must contain an edge from  $A(e)$ . If  $M'$  is a set of edges containing exactly one edge from  $A(e)$  for each such edge  $e$  then  $A(M') = E(G)$ . Thus it follows that  $|M|$  is the number of stems in  $G$  or  $G \cong K_2$  and thus  $G$  is equimatchable.  $\square$

**Corollary 1** *Let  $G$  be a connected equimatchable graph with girth  $g(G) \geq 5$ . If  $G$  has a stem, then  $G \in \mathcal{F}$ .*

**Proof.** The statement is trivially true if  $G \cong K_2$ . If  $s$  is a stem in  $G$  and  $s, v_1, v_2, \dots, v_{2k}$ ,  $k \geq 1$ , is a path in  $G$  then Lemma 5 implies that  $v_2, v_4, \dots, v_{2k}$  are stems and by Lemma 3  $v_1, v_3, \dots, v_{2k-1}$  are not stems. Thus it follows that  $G \in \mathcal{F}$ .  $\square$

**Corollary 2** *Let  $G$  be a connected equimatchable graph with girth  $g(G) \geq 8$ . Then  $G \in \mathcal{F}$ .*

**Proof.** By Lemma 4 the graph  $G$  must contain a stem, and thus Corollary 1 implies that  $G \in \mathcal{F}$ .  $\square$

### 3 Main Result

By using the results from Section 2 we obtain a simple characterization of the equimatchable graphs with girth at least five.

**Theorem 1** *Let  $G$  be a connected equimatchable graph with girth  $g(G) \geq 5$ . Then  $G \in \mathcal{F} \cup \{C_5, C_7\}$ .*

**Proof.** Assume that there is a connected equimatchable graph with girth  $g(G) \geq 5$  not contained in  $\mathcal{F} \cup \{C_5, C_7\}$ . Let  $G$  be such a graph of minimum size. It follows from Corollary 2 that  $g(G) \leq 7$  and by Corollary 1 it can be assumed that  $G$  does not have a stem. Assume that  $g(G) = 7$  and let  $C$  be a 7-cycle in  $G$ . Since  $G \not\cong C_7$  there must be a vertex  $u \in V(C)$

such that  $u$  is adjacent to a vertex  $v \notin V(C)$ . For each vertex  $z \in N[v] \setminus u$  let  $e_z$  be an edge incident with  $z$  and not incident with  $v$  and let  $M$  be all of these edges. Since  $g(G) = 7$  the set  $M$  is a matching and the vertex  $u$  is a stem in  $G - A(M)$ . Observe that  $C$  is still a cycle in the equimatchable graph  $G - A(M)$ . Since  $|E(G - A(M))| < |E(G)|$  a contradiction is obtained and we may assume that  $g(G) < 7$ .

Now assume that  $g(G) = 6$  and let  $C$  be a 6-cycle in  $G$ . Since  $C_6$  is not equimatchable there must be a vertex  $u \in V(C)$  such that  $u$  is adjacent to a vertex  $v \notin V(C)$ . Let  $z$  be a vertex from  $V(C)$  adjacent to  $u$ . For each vertex from  $x \in N(\{u, z\}) \setminus V(C)$  let  $e_x$  be an edge incident with  $x$  and not incident with  $u$  or  $z$ . If  $M$  denotes all of these edges then  $M$  is a matching in  $G$ . Thus since the girth is 6 it can be observed that  $G - A(M)$  has a component that contains  $C$  and two adjacent vertices  $u$  and  $v$  where  $\deg(u) = \deg(z) = 2$ . Since this component is not contained in  $\mathcal{F} \cup \{C_5, C_7\}$  but is equimatchable a contradiction is reached, and therefore we may assume that  $g(G) = 5$ .

Let  $C$  be a 5-cycle in  $G$ . Since  $G \not\cong C_5$  there must be a vertex  $u \in V(C)$  such that  $u$  is adjacent to a vertex  $v \notin V(C)$ . By considering an edge  $e$  not incident with a vertex from  $V(C)$  it follows that  $G - A(e)$  has a component containing  $C$  and if the component is not isomorphic to  $C$  then we have a contradiction with the choice of  $G$ . It follows that  $G$  must be the graph obtained from a  $C_5 : v_1, v_2, \dots, v_5, v_1$  by adding two vertices  $x, y$  and the edges  $v_1x, xy$  and  $yv_3$ . Since this graph is not equimatchable a contradiction is obtained.  $\square$

Theorem 1 along with Lemmas 2 and 6 complete the characterization.

We conclude by observing that it is easy to verify that the equimatchable graphs of girth 5 or more are also in the class A collection but considering the equimatchable graph  $K_{3,2}$  we see that once the girth is 4 the class A ones are a proper subcollection.

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